Non-typical points for β-shifts

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Abstract

We study sets of nontypical points under the map $f_\beta \mapsto \beta x$ mod 1, for non-integer β and extend our results from [2] in several directions. In particular we prove that sets of points whose forward orbit avoid certain Cantor sets, and set of points for which ergodic averages diverge, have large intersection properties. We remove the technical condition $\beta > 1.541$ found in [2].

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1 β-shifts

Let $[x]$ denote the integer part of the real number x, and let $\lfloor x \rfloor$ denote the largest integer strictly smaller than x. Let $\beta > 1$. For any $x \in [0,1]$ we associate the sequence $d(x,\beta) = (d(x,\beta)_n)_{n=0}^\infty \in \{0,1,\ldots,\lfloor\beta\rfloor\}^{\mathbb{N}}$ defined by

$$d(x,\beta)_n := [\beta f_\beta^n(x)],$$

where $f_\beta(x) = \beta x \pmod 1$. The closure, with respect to the product topology, of the set

$$\{\, d(x,\beta) : x \in [0,1) \,\}$$

is denoted by S_β and it is called the β-shift. We will denote the set of all finite words occuring in S_β by S_β^*. The sets S_β and S_β^* are invariant under the left-shift $\sigma \colon (i_n)_{n=0}^\infty \mapsto (i_{n+1})_{n=0}^\infty$ and the map $d(\cdot,\beta) \colon x \mapsto d(x,\beta)$ satisfies the equality $\sigma^n(d(x,\beta)) = d(f_\beta^n(x),\beta)$. If we order S_β with the lexicographical ordering then the map $d(\cdot,\beta)$ is one-to-one and monotone increasing. Let $d_-(1,\beta)$ be the limit in the product topology of $d(x,\beta)$ as x approaches 1 from below. Then the subshift S_β satisfies

$$S_\beta = \{\, (j_k)_{k=0}^\infty : \sigma^n(j_k)_{k=0}^\infty \leq d_-(1,\beta) \ \forall n \,\}. \tag{1}$$

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Note that $d_-(1,\beta) = d(1,\beta)$ if and only if $d(1,\beta)$ contains infinitely many non-zero digits.

Parry proved in [5] that the map $\beta \mapsto d(1,\beta)$ is monotone increasing and injective. For a sequence $(j_k)_{k=0}^{\infty}$ there is a $\beta > 1$ such that $(j_k)_{k=0}^{\infty} = d(1,\beta)$ if and only if $\sigma^n((j_k)_{k=0}^{\infty}) < (j_k)_{k=0}^{\infty}$ for every $n > 0$. The number β is then the unique positive solution of the equation

$$1 = \sum_{k=0}^{\infty} \frac{d_k(1,\beta)}{x^{k+1}}.$$

One observes that the fact that the map $\beta \mapsto d(1,\beta)$ is monotone increasing and injective together with (1) imply that $S_{\beta_1} \subseteq S_{\beta_2}$ holds if and only if $\beta_1 \leq \beta_2$.

If $x \in [0,1]$ then

$$x = \sum_{k=0}^{\infty} \frac{d_k(x,\beta)}{\beta^{k+1}}.$$

This formula can be seen as an expansion of x in the non-integer base β, and thereby generalises the ordinary expansion in integer bases.

We let π_β be the map $\pi_\beta \colon S_\beta \to [0,1)$ defined by

$$\pi_\beta \colon (i_k)_{k=0}^{\infty} \quad \mapsto \quad \sum_{k=0}^{\infty} \frac{i_k}{\beta^{k+1}}.$$

Hence, $\pi_\beta(d(x,\beta)) = x$ holds for any $x \in [0,1)$ and $\beta > 1$.

We define cylinder sets as

$$[i_0 \cdots i_{n-1}] := \{\, (j_k)_{k=0}^{\infty} \in S_\beta : i_k = j_k,\ 0 \leq k < n \,\},$$

and say that n is the generation of the cylinder $[i_0 \cdots i_{n-1}]$. We will also call the half-open interval $\pi_\beta([i_0 \cdots i_{n-1}])$ a cylinder of generation n. The set $[i_0 \cdots i_{n-2}]$ will be called the parent cylinder of $[i_0 \cdots i_{n-1}]$.

Note that if $d(1,\beta)$ has only finitely many non-zero digits, then S_β is a subshift of finite type, so there is a constant $C > 0$ such that

$$C\beta^{-n} \leq |\pi_\beta([i_0 \cdots i_{n-1}])| \leq \beta^{-n}. \tag{2}$$

2 Transversality and large intersection classes

In [1], Falconer defined $\mathcal{G}^s$, $0 < s \leq n$, to be the class of G_δ sets F in $\mathbb{R}^n$ such that $\dim_H(\cap_{i=1}^{\infty} f_i(F)) \geq s$ for all sequences of similarity transformations $(f_i)_{i=1}^{\infty}$. He characterised $\mathcal{G}^s$ in several equivalent ways and proved among other things that countable intersections of sets in $\mathcal{G}^s$ are also in $\mathcal{G}^s$.

In [2], the following approximation theorem was proven, where $\mathcal{G}^s$ are restrictions of Falconer's classes to the unit interval.

Theorem 1. *Let $\beta \in (1.541, 2)$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n \in (1.541, \beta)$ for all n, such that $\beta_n \to \beta$ as $n \to \infty$. Assume that $E \subset S_\beta$ and $\pi_{\beta_n}(E \cap S_{\beta_n})$ is in the class $\mathcal{G}^s$ for all n. If F is a G_δ set such that $F \supset \pi_\beta(E)$, then F is also in the class $\mathcal{G}^s$.*

When expanding a number x in base $\beta > 1$ as $d(x, \beta) = (x_k)_{k=0}^{\infty}$, one can consider how often a given word $y_1 \ldots y_m$ occurs. If the expression

$$\frac{\#\{i \in \{0, \ldots, n-1\} : x_i \ldots x_{i+m-1} = y_1 \ldots y_m\}}{n}$$

converges as $n \to \infty$, it gives an asymptotic frequency of the occurence of the word $y_1 \ldots y_m$ in the expansion of x to the base β. Theorem 1 was used in [2] to prove the following.

Proposition 1. *For any sequence of bases $(\beta_n)_{n=1}^{\infty}$, such that $\beta_n \in (1.541, 2)$ for all n, the set of points for which the frequency of any finite word does not converge in the expansion to any of these bases, has Hausdorff dimension 1.*

The reason for the condition $\beta \in (1.541, 2)$ in Theorem 1 and Proposition 1 is that we needed some estimates on the map

$$\sum_{k=0}^{\infty} \frac{a_k - b_k}{\beta_1^k} \mapsto \sum_{k=0}^{\infty} \frac{a_k - b_k}{\beta_2^k}, \quad (a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_{\beta_1}, \tag{3}$$

when $\beta_1 < \beta_2$, provided by the following transversality lemma by Solomyak [7].

Lemma 1. *Let $x_0 < 0.649$. There exists a constant $\delta > 0$ such that if $x \in [0, x_0]$ then*

$$|g(x)| < \delta \quad \Longrightarrow \quad g'(x) < -\delta$$

holds for any function of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k, \text{ where } a_k \in \{-1, 0, 1\}. \tag{4}$$

The condition $x_0 < 0.649$ in Lemma 1 introduces the condition $\beta > 1/0.649$ or for simplicity $\beta > 1.541$. But, when studying the map (3), the coefficients in the power series (4) will not be free to take values in $\{-1, 0, 1\}$, they will be the difference of two sequences from S_β. This allows us to remove the condition $x_0 < 0.649$.

Lemma 2. *Let $\beta > 1$. There exists a constant $\delta > 0$ such that if $x \in [0, 1/\beta]$ then*

$$|g(x)| < \delta \quad \Longrightarrow \quad g'(x) < -\delta$$

holds for any function of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k, \quad \text{where } (a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_\beta.$$

Proof. Assume that no such δ exists. Then there is a sequence g_n of power series and a sequence of numbers $x_n \in [0, 1/\beta]$, such that $\lim_{n\to\infty} g_n(x_n) = 0$ and $\liminf_{n\to\infty} g_n'(x_n) \geq 0$.

We can take a subsequence such that g_n converges termwise to a series $g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k$, with $(a_1, a_2 \ldots), (b_1, b_2 \ldots) \in S_\beta$, and x_n converges to some number x_0. Clearly, $g(x_0) = 0$ and $g'(x_0) \geq 0$, so $x_0 \neq 0$.

Let $\beta_0 = 1/x_0 \geq \beta$. Then $(a_1, a_2 \dots), (b_1, b_2 \dots) \in S_{\beta_0}$ and $g(x_0) = 0$ implies that

$$\pi_{\beta_0}(a_1, a_2, \dots) - \pi_{\beta_0}(b_1, b_2, \dots) = \sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} - \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = -1.$$

Since both sums are in $[0, 1]$, we conclude that

$$\sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = 1.$$

We must therefore have $(a_1, a_2, \dots) = (0, 0, \dots)$. This implies that $g'(x) < 0$ for all $x \in (0, 1/\beta]$, contradicting the fact that $g'(x_0) \geq 0$. $\square$

Replacing Lemma 1 by Lemma 2 in the proofs of [2], we immediately get the following improved versions of Theorem 1 and Proposition 1. Note that allowing $\beta > 1$ instead of $\beta \in (1, 2)$ only affects notation slightly by adding new symbols to the shift space S_β. The proofs in [2] go through almost verbatim. Also the result from [3], which is used in [2] to prove Proposition 1, is easily extended from $\beta \in (1, 2)$ to $\beta > 1$.

Theorem 2. *Let $\beta > 1$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n < \beta$ for all n, such that $\beta_n \to \beta$ as $n \to \infty$. Assume that $E \subset S_\beta$ and $\pi_{\beta_n}(E \cap S_{\beta_n})$ is in the class $\mathcal{G}^s$ for all n. If F is a G_δ set such that $F \supset \pi_\beta(E)$, then F is also in the class $\mathcal{G}^s$.*

Proposition 2. *For any sequence of bases $(\beta_n)_{n=1}^{\infty}$, such that $\beta_n > 1$ for all n, the set of points for which the frequency of any finite word does not converge in the expansion to any of these bases, has Hausdorff dimension 1.*

3 Schmidt games and avoiding Cantor sets

In [6], Schmidt introduced a set-theoretic game which can be seen as a metric version of the Banach–Mazur game (see for example [4]). We present here a modified version of Schmidt's game that was used in [2].

Consider the unit interval $[0, 1]$ with the usual metric and a set $E \subset [0, 1]$. Two players, Black and White, play the game in $[0, 1]$ with two parameters $0 < \alpha, \gamma < 1$ according to the following rules:

> *In the initial step* Black chooses a closed interval $B_0 \subset [0, 1]$.
>
> *Then the following step is repeated.* At step k White chooses a closed interval $W_k \subset B_k$ such that $|W_k| \geq \alpha |B_k|$. Then Black chooses a closed interval $B_{k+1} \subset W_k$ such that $|B_{k+1}| \geq \gamma |W_k|$.

We say that E is (α, γ)-winning if there is a strategy that White can use to make sure that $\bigcap_k W_k \subset E$, and α-winning if this holds for all γ. As was shown in [2], the following proposition easily follows from the methods in [6].

Proposition 3.

a. *If E is α-winning for $\alpha = \alpha_0$, then E is α-winning for all $\alpha \leq \alpha_0$.*

b. If E_i is α-winning for $i = 1, 2, 3, \ldots$, then $\cap_{i=1}^{\infty} E_i$ is also α-winning.

c. If E is α-winning, then the Hausdorff dimension of E is 1.

In [2] the following proposition was proven.

Proposition 4. *For any $\beta \in (1, 2)$ and any $x \in [0, 1]$,*

$$G_\beta(x) = \Big\{ y \in [0,1] : y \notin \overline{\bigcup_{n=1}^{\infty} f_\beta^n(y)} \Big\}.$$

is α-winning for any $\alpha \leq 1/16$.

The set $G_\beta(x)$ consists of points for which the forward orbit under f_β is bounded away from x. We will prove the following proposition which shows that we can avoid entire Cantor sets instead of just single points.

Proposition 5. *Let $\beta > 1$ and let $\Sigma_A \subset S_\beta$ be a subshift of finite type, such that there is a finite word $i_1 \ldots i_n$ from $S_\beta \setminus \Sigma_A$. Then there exist $\alpha > 0$ such that*

$$G_{f_\beta}(\pi_\beta(\Sigma_A)) = \Big\{ y \in [0,1) : \pi_\beta(\Sigma_A) \ \cap \overline{\bigcup_{n=1}^{\infty} f_\beta^n(y)} = \emptyset \Big\}$$

is α-winning.

A quick look at Proposition 3 gives us the following corollary.

Corollary 1. *Let $N \in \mathbb{N}$, $\beta_1, \ldots, \beta_N > 1$ and for each $1 \leq n \leq N$, let $\Sigma_{A_n} \subset S_{\beta_n}$ such that there is a finite word $i_1 \ldots i_{k_n}$ from $S_{\beta_n} \setminus \Sigma_{A_n}$. The the set*

$$\bigcap_{n=1}^{N} G_{f_{\beta_n}}(\pi_\beta(\Sigma_{A_n}))$$

has Hausdorff dimension 1.

The reason that N in Corollary 1 must be finite is that α_0 from Proposition 5 will depend on β and Σ_A. When taking intersections we need a uniform α for which the sets are α-winning, to be able to say anything about the intersection. See Remark 2 at the end of the paper for an estimate of α.

Before giving the proof of Proposition 5, we note that if S_β is a subshift of finite type, then Proposition 5 is easy. Indeed, by (2) we have good control over the size of each cylinder. So, it is not hard to see that there is an $\alpha_0 > 0$ such that each time White plays he can introduce the word $i_1 \ldots i_n$. Again by (2) this implies that the word $i_1 \ldots i_n$ occurs regularly in $\{y\} = \cap_k W_k$, and this means that $\bigcup_{n=1}^{\infty} f_\beta^n(y)$ is bounded away from $\pi_\beta(\Sigma_A)$. Hence Proposition 5 need only be proved in the case when S_β is not of finite type.

The case when S_β is not of finite type is much more difficult, since we have no uniform lower bound on the size of cylinders, such as (2). The key step in proving Proposition 4 was the following theorem from [2]. It will be used in the proof of Proposition 5.

Theorem 3. *Let $\beta \in (1, 2)$ and let $(\beta_n)_{n=1}^{\infty}$ be any sequence with $\beta_n \in (1, \beta)$ for all n such that $\beta_n \to \beta$ as $n \to \infty$. Let also $E \subset S_\beta$ and $\alpha \in (0, 1)$. If $\pi_{\beta_n}(E \cap S_{\beta_n})$ is α-winning for $\alpha = \alpha_0$ for all n, then $\pi_\beta(E)$ is α-winning for any $\alpha \leq \min\{\frac{1}{16}, \frac{\alpha_0}{4}\}$.*

Remark 1. *The condition $\beta \in (1,2)$ in Theorem 3 comes from the fact that in [2], we chose to work with $\beta < 2$ to simplify notation. It is not difficult to extend the proof of Theorem 3 to hold for all $\beta > 1$.*

The only place in which $\beta \in (1,2)$ was used is in what is called "An auxiliary strategy". There we use the fact that in any cylinder $\pi_\beta([i_0 \dots i_n])$, the player White needs at most a factor 2 to make sure that the game continues in $\pi_\beta([i_0 \dots i_n 0])$, thereby avoiding the cylinder $\pi_\beta([i_0 \dots i_n 1])$ which may have bad properties. If $\beta > 2$, a factor 2 is still enough for White to avoid the cylinder $\pi_\beta([i_0 \dots i_n \lfloor \beta \rfloor])$ which may have bad properties. The factor 2 is not enough for White to choose any other cylinder $\pi_\beta([i_0 \dots i_n k])$ in one move, but after a couple of moves, the game is already played in such a small set that at most two of these cylinders remain, so White can pick at least one of them. That is all what is needed for the strategy to work.

Proposition 5 follows from Theorem 3 and Remark 1 once we have proven the following proposition.

Proposition 6. *Let $\beta > 1$ such that S_β is not of finite type and let $\Sigma_A \subset S_\beta$ be a subshift of finite type. Then there exist $\alpha > 0$ and $\beta_0 < \beta$ such that*

$$G_{\beta'}(\pi_{\beta'}(\Sigma_A)) = \Big\{ y \in [0,1) : \pi_{\beta'}(\Sigma_A) \cap \overline{\bigcup_{n=0}^{\infty} f_{\beta'}^n(y)} = \emptyset \Big\}$$

is α-winning for any $\beta' \in [\beta_0, \beta]$ such that $S_{\beta'}$ is of finite type.

To prove Proposition 6, we need some lemmata.

Lemma 3. *Let $\beta > 1$ and let $i_1 \dots i_n$ be a finite word in S_β^* such that we have $i_1 \dots i_n j_1 \dots j_m \in S_\beta^*$ for all finite words $j_1 \dots j_m \in S_\beta^*$. Then $|\pi([i_1 \dots i_n])| = \beta^{-n}$ and also $|\pi_\beta([i_1 \dots i_n j_1 \dots j_m])| = \beta^{-n} |\pi_\beta([j_1 \dots j_m])|$ for all finite words $j_1 \dots j_m \in S_\beta^*$.*

Proof. It is clear that $\sigma^n([i_1 \dots i_n]) = S_\beta$, so $f_\beta^n(\pi_\beta([i_1 \dots i_n])) = [0,1)$, where f_β^n is just the scaling $x \mapsto \beta^n x$ on $\pi_\beta([i_1 \dots i_n])$. Thus, $\pi_\beta([i_1 \dots i_n])$ is just a smaller copy of $[0,1)$. □

Lemma 4. *Let $\beta > 1$, $M \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $(d(1,\beta)_n)_{n=0}^M 0^k 1 \in S_\beta^*$. If $\beta_0 \in (1,\beta)$ is such that $(d(1,\beta)_n)_{n=0}^M 0^k 1 \in S_{\beta_0}^*$, then for all $i_1 \dots i_n \in S_{\beta_0}^*$ such that $M = \max\{\, m : i_{n-m} \dots i_n = (d(1,\beta)_n)_{n=0}^m \,\}$, it holds that*

$$|\pi_{\beta'}([i_1 \dots i_n])| \geq \beta^{-(n+k+1)}, \textit{ for all } \beta' \in [\beta_0, \beta].$$

Proof. Let $\beta' \in [\beta_0, \beta]$. From (1) and the maximality of M we conclude that $i_1 \dots i_{n-M} j_1 \dots j_m \in S_{\beta'}^*$ for all $j_1 \dots j_m \in S_{\beta'}^*$. From Lemma 3 we then get $|\pi_\beta([i_1 \dots i_n])| \geq |\pi_\beta([i_1 \dots i_n 0^{k+1}])| = \beta^{-(n+k+1)}$. □

Lemma 5. *Let $\beta > 1$ and $M \in \mathbb{N}$. There exist $\epsilon > 0$ and $\beta_0 < \beta$ such that for any $\beta' \in [\beta_0, \beta]$ and for any interval $I \subset [0,1]$, there exists a cylinder $\pi_{\beta'}([i_0 \dots i_n])$ such that $\max\{m : i_{n-m} \dots i_n = (d(1,\beta)_n)_{n=0}^m\} \geq M$ for which $|\pi_{\beta'}([i_0 \dots i_n]) \cap I| > \epsilon |I|$. Moreover, if $S_{\beta'}$ is of finite type, then $|\pi_{\beta'}([i_0 \dots i_n]) \cap I| > \sigma_{\beta'} |\pi_{\beta'}([i_0 \dots i_n])|$, where $\sigma_{\beta'} > 0$ is independent of I.*

Proof. Let $\beta' \in [\beta_0, \beta]$ as in Lemma 4 and let $I \subset [0,1]$ be an interval. Note that all cylinders in this proof will be with respect to $S_{\beta'}$. Let n be the smallest generation for which there is a cylinder contained in I. Let $\pi_{\beta'}([i_0 \dots i_{n-1}])$ be one of these generation n cylinders in I. By the minimality of n we know that the parent cylinder, $\pi_{\beta'}([i_0 \dots i_{n-2}])$ covers at least one endpoint of I. If $\pi_{\beta'}([i_0 \dots i_{n-2}])$ does not cover I, let m be the smallest generation for which there is a cylinder contained in $I \setminus \pi_{\beta'}([i_0 \dots i_{n-2}])$. Let $\pi_{\beta'}([j_0 \dots j_{m-1}])$ be one of these generation m cylinders. By the minimality of m we know that the parent cylinder, $\pi_{\beta'}([j_0 \dots j_{m-2}])$ covers the other endpoint of I.

Together, $\pi_{\beta'}([i_0 \dots i_{n-2}])$ and $\pi_{\beta'}([j_0 \dots j_{m-2}])$ cover I. Indeed, if not, then there is a smallest generation l for which there is a cylinder $\pi_{\beta'}([k_0 \dots k_{l-1}])$ between $\pi_{\beta'}([i_0 \dots i_{n-2}])$ and $\pi_{\beta'}([j_0 \dots j_{m-2}])$. Consider its parent cylinder $\pi_{\beta'}([k_0 \dots k_{l-2}])$. If $\pi_{\beta'}([k_0 \dots k_{l-2}])$ would intersect one of $\pi_{\beta'}([i_0 \dots i_{n-2}])$ and $\pi_{\beta'}([j_0 \dots j_{m-2}])$, then it would have to contain it. But this is impossible since the minimality of n and m implies $l \geq n, m$. Thus, $\pi_{\beta'}([k_0 \dots k_{l-2}])$ is also between $\pi_{\beta'}([i_0 \dots i_{n-2}])$ and $\pi_{\beta'}([j_0 \dots j_{m-2}])$, which contradicts the minimality of l.

Consider the one of $\pi_{\beta'}([i_0 \dots i_{n-2}])$ and $\pi_{\beta'}([j_0 \dots j_{m-2}])$ that covers at least half of I. Let us assume it is $\pi_{\beta'}([i_0 \dots i_{n-2}])$ but it makes no difference for the argument.

If $\max\{\, m : i_{n-m-2} \dots i_{n-2} = (d(1,\beta)_n)_{n=0}^m \,\} \geq M$, then we can choose the set $\pi_{\beta'}([i_0 \dots i_{n-2}]) \cap I$ as long as $\epsilon \leq 1/2$ and we are done with the first claim. The second claim, that $|\pi_{\beta'}([i_0 \dots i_{n-2}]) \cap I| > \sigma_{\beta'} |\pi_{\beta'}([i_0 \dots i_{n-2}])|$ follows from the fact that $|\pi_{\beta'}([i_0 \dots i_{n-1}]) \subset I$ and (2), since $S_{\beta'}$ is of finite type.

Assume instead that $\max\{m : i_{n-m-2} \dots i_{n-2} = (d(1,\beta)_n)_{n=0}^m\} = N < M$. Then $i_0 \dots i_{n-2} (d(1,\beta)_n)_{M-N}^M \in S_{\beta'}^*$ by (1). By Lemma 4 there is a k that only depends on β and M such that

$$|\pi_{\beta'}([i_0 \dots i_{n-2}(d(1,\beta)_n)_{M-N}^M])| \geq \beta^{-(n+M+k)} \geq \beta^{-(M+k+1)} |\pi_{\beta'}([i_0 \dots i_{n-2}])|.$$

We conclude that if $\epsilon \leq \beta^{-(M+k+1)}/2$, then we can choose the cylinder $\pi_{\beta'}([i_0 \dots i_{n-2}(d(1,\beta)_n)_{M-N}^M]) \subset I$. This ensures the truth of both claims and we are done. □

We are now ready to prove Proposition 6.

Proof of Proposition 6. Note that since Σ_A is of finite type while S_β is not, there is an $M > 1$ such that $(d(1,\beta)_n)_{n=0}^M$ is not allowed in Σ_A. For this M choose ϵ and β_0 as in Lemma 5. Let $\beta' \in [\beta_0, \beta]$ such that $S_{\beta'}$ is of finite type, let $\gamma > 0$ and $\alpha = \epsilon/2$.

Assume that Black has chosen his first interval B_0. We will construct a strategy that White can use to make sure that $\cap_k W_k = \{x\} \subset G_{\beta'}(\Sigma_A)$, or equivalently that $(f_\beta^n(x))_{n=0}^\infty$ is bounded away from $\pi_{\beta'}(\Sigma_A)$.

Each time Black has chosen an interval B_k, Lemma 5 ensures that White can choose $W_k \subset \pi_{\beta'}[i_1 \dots i_n] \cap B_k$, where $\max\{\, m : i_{n-m} \dots i_n = (d(1,\beta)_n)_{n=0}^m \,\} \geq M$ and $|W_k| \geq \sigma(\beta') |\pi_{\beta'}[i_1 \dots i_n]|$. Since $\beta' < \beta$, there is an N such that $(d(1,\beta)_n)_{n=0}^N \notin S_{\beta'}$, so $M \leq \max\{\, m : i_{n-m} \dots i_n = (d(1,\beta)_n)_{n=0}^m \,\} \leq N$.

If White plays like this, it ensures that the sequence $d(x, \beta')$ contains the word $(d(1,\beta)_n)_{n=0}^M$ regularly. Thus, $(f_\beta^n(x))_{n=0}^\infty$ is always in a cylinder outside $\pi_{\beta'}(\Sigma_A)$. If $(f_\beta^n(x))_{n=0}^\infty$ would be bounded away from the endpoints of these cylinders, then $(f_\beta^n(x))_{n=0}^\infty$ would be bounded away from $\pi_{\beta'}(\Sigma_A)$. But $\alpha = \epsilon/2$,

so there is a factor 2 left after placing W_k in $\pi_{\beta'}[i_1 \dots i_n] \cap B_k$. White can place W_k in the middle of $\pi_{\beta'}[i_1 \dots i_n] \cap B_k$, thereby avoiding the endpoints.

We conclude that $G_{\beta'}(\Sigma_A)$ is α-winning and we are done. $\square$

Remark 2. *The α in Proposition 6 can be extracted quite easily from the proofs. Let M be such that $(d_k(1,\beta))_{k=1}^M$ is not at word in Σ_A. Take k such that $(d_j(1,\beta))_{j=0}^M 0^k 1 < d(1,\beta)$. Then $\alpha = \beta^{-(M+k+1)}/4$ is small enough. It follows that in Proposition 5, $\alpha = \beta^{-(M+k+1)}/16$ is small enough. Note that these values for α are not optimal, but they make it possible to extend Corollary 1 to infinite intersections, for some cases.*

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